THE GALAXIES OF NONSTANDARD ENLARGEMENTS OF INFINITE AND TRANSFINITE GRAPHS

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Abstract — The galaxies of the nonstandard enlargements of conventionally infinite graphs as well as of transfinite graphs are defined, analyzed, and illustrated by some examples. It is then shown that any such enlargement either has exactly one galaxy, its principal one, or it has infinitely many galaxies. In the latter case, the galaxies are partially ordered by there "closeness" to the principal galaxy. If an enlargement has a galaxy different from its principal galaxy, then it has a two-way infinite sequence of galaxies that are totally ordered according to that "closeness" property. There may be many such totally ordered seqences.

Key Words: Nonstandard graphs, enlargements of graphs, transfinite graphs, galaxies in nonstandard graphs, graphical galaxies.

1   Introduction

In this work we extend the idea of galaxies in the hyperreal line $^*\!R$ to nonstandard enlargements of conventionally infinite graphs and also of transfinite graphs. Since graphs have structures much different from that of the real line R, the enlargements of graphs have properties not possessed by $^*\!R$. The graphical galaxies of those enlargements comprise one aspect of that distinctive complexity. We will show that that any such enlargement has either one galaxy or infinitely many of them. Moreover, just as $^*\!R$ contains images of the real numbers, called the standard hyperreals, as well as hyperreals that are nonstandard, so too may the enlargement $^*\!G$ of a graph G contain "hypernodes," some of which are images of nodes of G and others of which are nonstandard hypernodes. In addition, there are "hyperbranches" incident to pairs of hypernodes; some of these hyperbranches are images

of branches of G, but there may be others that are not.

The galaxies graphically partition ${}^*G$ in the sense the every hypernode belongs to exactly one galaxy, and so too does every hyperbranch. There is a unique galaxy, which we refer to as the "principal galaxy," that contains the standard hypernodes and possibly nonstandard hypernodes as well. In the event that there are infinitely many galaxies, those galaxies are partially ordered according to how "close" they are to the principal galaxy. In fact, if there is a galaxy different from the principal galaxy, then there is a two-way infinite sequence of galaxies that are totally ordered according to their "closeness" to the principal galaxy. There may be many such totally ordered sequences, but a galaxy in one such sequence may not be comparable to a galaxy in another sequence according to that "closeness" property.

We speak of "conventionally infinite" graphs to distinguish them from transfinite graphs of ranks 1 or higher [5, Chapter 2], [6, Chapter 2]. Sections 2 through 4 herein are devoted to the enlargements of conventionally infinite graphs. The results for such enlargements extend to enlargements of transfinite graphs, but in more complicated ways. We show this in Sections 5 through 11, but only for transfinite graphs of rank 1. Results for transfinite graphs of still higher ranks are obtained similarly but in still more complicated ways and with additional complexity in the symbols. For the sake of brevity, the latter results are not included herein, but they may be found in [7] as well as in the archive www.arxiv.org in the category "mathematics" under "A.H. Zemanian."

Our notations and terminology follow the usual conventions of nonstandard analysis. $\textbf{\textit{N}} = \{0, 1, 2, \ldots\}$ is the set of natural numbers, and ${}^*\textbf{\textit{N}}$ is the set of hypernaturals. The standard hypernaturals are (i.e., can be identified with) the natural numbers. Also, $\langle a_n \rangle$ or $\langle a_n : n \in \textbf{\textit{N}} \rangle$ or $\langle a_0, a_1, a_2, \ldots \rangle$ denotes a sequence whose elements can be members of any set, such as the set X of nodes in a conventional graph $G = \{X, B\}$, where B is the set of branches, a branch being a two-element set of nodes. On the other hand, $[a_n]$ denotes an equivalence class of sequences, where two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ are taken to be equivalent if $\{n : a_n = b_n\} \in \mathcal{F}$, where $\mathcal{F}$ is any chosen and fixed free ultrafilter.[1] The a_n appearing in $[a_n]$ are understood to be the elements of any one of the sequences in the equivalence class.

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[1] Also called a nonprincipal ultrafilter.

At times, we will use the more specific notation $[\langle a_0, a_1, a_2, \ldots \rangle]$. More generally, we adhere to the notations and terminology appearing in [3].

The ordinals are denoted in the usual way: ω is the first transfinite ordinal. With $\tau \in \boldsymbol{N}$, the product $\omega \cdot \tau$ is the sum of τ summands, each being ω.

2   The Nonstandard Enlargement of a Graph

Throughout Sections 2 to 4, we assume that the conventionally infinite graph G is connected and has infinitely many nodes. The definition of a nonstandard graph that we use herein is given in [6, Section 8.1], a special case of which is the "enlargement" of a graph G.

Let us define the *enlargement* $^*G$ of G here as well in order to remove any need for referring to [6]. $G = \{X, B\}$ is now taken to be a conventional connected graph having an infinite set X of nodes and therefore an infinite set of branches as well, each branch being a two-element set of nodes. Thus, there are no parallel branches (i.e., multiple branches). $\mathcal{F}$ will denote a chosen and fixed free ultrafilter. $\mathbf{x} = [x_n]$ denotes an equivalence class of sequences of nodes as stated in the Introduction. $\mathbf{x}$ will be called a *hypernode*.[2] Thus, the set of all sequences of nodes from G is partitioned into hypernodes. $^*X$ denotes the set of hypernodes. If all the elements of one of the representative sequences $\langle x_n \rangle$ for a hypernode $\mathbf{x} = [x_n]$ are the same node (i.e., $x_n = x$ for all n), then $\mathbf{x} = [x]$ can be identified with x; in this case, $\mathbf{x}$ is called a *standard hypernode*. Otherwise, $\mathbf{x} = [x_n]$ is called a *nonstandard hypernode*.

We turn now to the definition of a "hyperbranch." Let $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ be two hypernodes. Also, let $\mathbf{b} = [\{x_n, y_n\}]$, where $\langle \{x_n, y_n\} \rangle$ is a sequence of pairs of nodes from G such that, for almost all n, $\{x_n, y_n\}$ is a branch in G; that is, $\{n : \{x_n, y_n\} \in B\} \in \mathcal{F}$. It can be shown [6, page 155] that this definition is independent of the representative sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ chosen of $\mathbf{x}$ and $\mathbf{y}$ respectively and that we truly have an equivalence relation for the set of all sequences of branches from G. We let $\mathbf{b} = [\{x_n, y_n\}]$ denote such an equivalence class and will call it a *hyperbranch*; we write $\mathbf{b} = \{\mathbf{x}, \mathbf{y}\}$. Also, $^*B$ will denote the set of all hyperbranches. If $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ are standard hypernodes, then $\mathbf{b} = [\{x, y\}]$

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[2]Our terminology should not be confused with that of a hypergraph—an entirely different concept [2].

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is called a *standard hyperbranch*. Otherwise, **b** is called a *nonstandard hyperbranch*.

Finally, the pair ${}^*G = \{{}^*X, {}^*B\}$ denotes the *enlargement* of G. It is a special case of a nonstandard graph, as defined in [6, page 155].[3]

3   Distances and Galaxies in Enlarged Graphs

The *length* $|P_{x,y}|$ of any path $P_{x,y}$ connecting two nodes x and y in a graph G is the number of branches in $P_{x,y}$. The *distance* $d(x,y)$ between x and y is $d(x,y) = \min\{|P_{x,y}|\}$, where the minimum is taken over all paths terminating at x and y. In the trivial case, $d(x,x) = 0$. d satisfies the triangle inequality, namely, for any three nodes x, y, and z in G, $d(x,y) \le d(x,z) + d(z,y)$. In fact, d satisfies the other metric axioms, too, and the set X of nodes in G along with d is a metric space.

The metric d can be extended into an internal function **d** mapping the Cartesian product ${}^*X \times {}^*X$ into the set of hypernaturals ${}^*N$ as follows: For any $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ in ${}^*X$, **d** is defined by

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) \;=\; [d(x_n, y_n)] \in {}^*N.$$

By the transfer principle, we have, for any three hypernodes **x**, **y**, and **z**,

$$\mathbf{d}(\mathbf{x}, \mathbf{z}) \;\le\; \mathbf{d}(\mathbf{x}, \mathbf{y}) + \mathbf{d}((\mathbf{y}, \mathbf{z}). \tag{1}$$

From the point of view of an ultrapower construction, this means that

$$\{n\colon d(x_n, z_n) \;\le\; d(x_n, y_n) + d(y_n, z_n\} \;\in\; \mathcal{F}.$$

The other metric axioms, such as $\mathbf{d}(\mathbf{x}, \mathbf{x}) = 0$, are obviously satisfied by **d**.

We define the "galaxies" of ${}^*G$ as nonstandard subgraphs of ${}^*G$ by first defining the "nodal galaxies." Two hypernodes $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ are taken to be in the same *nodal galaxy* $\dot{\Gamma}$ of ${}^*G$ if $\mathbf{d}(\mathbf{x}, \mathbf{y})$ is no greater that a standard hypernatural **k**, that is, if there exists a natural number $k \in N$ such that $\{n\colon d(x_n, y_n) \le k\} \in \mathcal{F}$. In this case, we say that **x** and **y** are *limitedly distant*, and we write $\mathbf{d}(\mathbf{x}, \mathbf{y}) \le \mathbf{k}$.

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[3]If G were a finite graph, then every hypernode (resp. hyperbranch) could be identified with a node (resp. branch)in G, and ${}^*G$ would be identified with G.

Let $N_{\mathbf{x},\mathbf{y}}$ be the set of all standard hypernaturals that are no less than $\mathbf{d}(\mathbf{x},\mathbf{y})$. $N_{\mathbf{x},\mathbf{y}}$ is a well-ordered set, and therefore it has a minimum $\mathbf{k}_{\mathbf{x},\mathbf{y}}$. So, we can say that $\mathbf{x}$ and $\mathbf{y}$ are in the same nodal galaxy $\dot{\Gamma}$ if $\mathbf{d}(\mathbf{x},\mathbf{y}) = \mathbf{k}_{\mathbf{x},\mathbf{y}}$.

Lemma 3.1. *The nodal galaxies partition the set $^*X$ of all hypernodes in $^*G$.*

Proof. The property of two hypernodes being limitedly distant is a binary relation on $^*X$ that is obviously reflexive and symmetric. Its transitivity follows directly from (1). Alternatively, we can use an ultrapower argument. Assume that $\mathbf{x} = [x_n]$ and $\mathbf{y} = [z_n]$ are in some nodal galaxy and that $\mathbf{y}$ and $\mathbf{z} = [z_n]$ are in some nodal galaxy; we want to show that those galaxies are the same. There exist two standard natural numbers k_1 and k_2 such that $N_{\mathbf{x},\mathbf{y}} = \{n : d(x_n, y_n) \le k_1\} \in \mathcal{F}$ and $N_{\mathbf{y},\mathbf{z}} = \{n : d(y_n, z_n) \le k_2\} \in \mathcal{F}$. Since $d(x_n, z_n) \le d(x_n, y_n) + d(y_n, z_n)$,

$$\{n : d(x_n, z_n) \le k_1 + k_2\} \supseteq N_{\mathbf{x},\mathbf{y}} \cap N_{\mathbf{y},\mathbf{z}} \in \mathcal{F}.$$

So, the left-hand side is a set in $\mathcal{F}$. Thus, $\mathbf{x}$ and $\mathbf{z}$ are limitedly distant, too, and $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are all in the same nodal galaxy. $\square$

We define a *galaxy* Γ of $^*G$ as a maximal nonstandard subgraph of $^*G$ whose hypernodes are all in the same nodal galaxy $\dot{\Gamma}$; that is, the hyperbranches of Γ corresponding to $\dot{\Gamma}$ are all those pairs $\{\mathbf{x}, \mathbf{y}\}$ such that $\mathbf{x}, \mathbf{y} \in \dot{\Gamma}$. We will say that a hypernode $\mathbf{x}$ is *in* Γ when $\mathbf{x} \in \dot{\Gamma}$ and that a hyperbranch $\{\mathbf{x}, \mathbf{y}\}$ is *in* Γ when $\mathbf{x}, \mathbf{y} \in \dot{\Gamma}$. It follows from Lemma 3.1 that the galaxies of $^*G$ partition $^*G$ in the sense of graphical partitioning (i.e., each hyperbranch is in one and only one galaxy).

The *principal galaxy* Γ_0 of $^*G$ is that unique galaxy, each of whose hypernodes is limitedly distant from some standard hypernode (and therefore from all standard hypernodes). All the nodes in G will be (i.e., can be identified with) standard hypernodes in Γ_0, but there may be nonstandard hypernodes in Γ_0 as well. The following examples illustrate this point.

Example 3.2. Consider the endless (i.e., two-way infinite) path:

$$P = \langle \ldots, x_{-1}, b_{-1}, x_0, b_0, x_1, b_1 \ldots \rangle$$

with nodes x_k and branches b_k, $k \in \mathbf{Z}$, $\mathbf{Z}$ being the set of integers. The enlargement $^*P$ of P has hypernodes, each being represented by $[x_{n_k}]$ where $\langle k_n \rangle$ is some sequence of integers.

Each hyperbranch is represented by $[\{x_{k_n}, x_{k_n+1}]$. The nodal galaxies are infinitely many because they correspond bijectively with the galaxies of the enlargement $^*\mathbf{Z}$ of $\mathbf{Z}$. Moreover, the principal galaxy Γ_0 of $^*P$ has only standard hypernodes and in fact is (i.e., can be identified with) P itself. Also, every galaxy is graphically isomorphic to Γ_0 and therefore to every other galaxy. $\square$

Example 3.3. Now, consider a one-ended path:

$$T = \langle x_0, b_0, x_1, b_1, x_2, b_2, \ldots \rangle$$

Each hypernode in the enlargement $^*T$ of T is represented by $[x_{k_n}]$, where $\langle k_n \rangle$ is some sequence of natural numbers. Thus, $^*T$ has a hypernode set $^*X$ that can be identified with the set $^*\mathbf{N}$ of hypernaturals. Hence, $^*T$ has an infinitely of galaxies, too. The principal galaxy Γ_0 of $^*T$ is the one-ended path T. However, any hypernode $\mathbf{x} = [x_{k_n}]$ in a galaxy Γ different from Γ_0 will be such that, for every $m \in \mathbf{N}$, $\{n\colon k_n > m\} \in \mathcal{F}$. Such a hypernode is adjacent both to $[x_{k_n+1}]$ and to $[x_{k_n-1}]$, where we are free to replace x_{k_n-1} by, say, x_0 whenever $k_n = 0$. (The set $\{n\colon k_n = 0\}$ will not be a member of $\mathcal{F}$ when $\mathbf{x} = [x_{k_n}]$ is in Γ.) Thus, $\mathbf{x} = [x_{k_n}] \in \Gamma$ has both a predecessor and a successor, which implies that Γ is graphically isomorphic to an endless path. In fact, all the galaxies other than Γ_0 are isomorphic to each other, being identifiable with an endless path. $\square$

Example 3.4. Consider next the grounded, one-way infinite ladder L of Figure 1. Now, for every $k \in \mathbf{N}$, $d(x_k, x_g) = d(x_k, x_{k+1}) = 1$, and, for every $k, l \in \mathbf{N}$ with $|k - l| > 1$, $d(x_k, x_l) = 2$. In this case, for every two hypernodes $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{d}(\mathbf{x}, \mathbf{y}) \le [2] = 2$. Thus, every two hypernodes are limitedly distant from each other, which means that $^*L$ has only one galaxy, its principal galaxy Γ_0. Now, Γ_0 has both standard and nonstandard hypernodes. $\square$

Example 3.5. Furthermore, consider the graph G obtained from L by appending a one-ended path P starting at x_g, but otherwise isolated from L, as shown in Figure 2. In this case, we again have an infinity of galaxies by virtue of the isolation of P from L. The principal galaxy Γ_0 has both standard and nonstandard hypernodes, its nonstandard hypernodes being due to L. All the other galaxies are graphically isomorphic to an endless path (as in Example 3.3) and thus to each other, but not to G and not to Γ_0. $\square$

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A subgraph G_s of G with the property that there exists a natural number k such that $d(x, y) \leq k$ for all pairs of nodes x, y in G_s will be called a *finitely dispersed* subgraph of G. Example 3.5 suggests that the structures of the galaxies other than Γ_0 do not depend upon any finitely dispersed subgraph of G. This is true in general because the nodes x_n in any representative $\langle x_n \rangle$ of any hypernode in a galaxy other than Γ_0 must lie outside any finitely dispersed subgraph of G for almost all n whatever be the choice of that finitely dispersed subgraph.

For instance, consider

Example 3.6. Let D_2 be the 2-dimensional grid; that is, we can represent D_2 by having its nodes at the lattice points (k, l) of the 2-dimensional plane, where $k, l \in \mathbf{Z}$ and with its branches being $\{(k, l), (k+1, l)\}$ and $\{(k, l), (k, l+1)\}$. So, the hypernodes of ${}^*D_2$ occur at ${}^*\mathbf{Z} \times {}^*\mathbf{Z}$. Under this representation, the principal nodal galaxy of ${}^*D_2$ will have its nodes at the lattice points of $\mathbf{Z} \times \mathbf{Z}$.

Next, let G be a connected graph obtained from D_2 by deleting or appending finitely many branches to D_2. So, outside a finitely dispersed subgraph of G, G is identical to D_2. Then the principal galaxy Γ_0 of ${}^*G$ is the same as (i.e., is graphically isomorphic to) G, but every other galaxy is the same as D_2. $\square$

In view of Examples 3.3 and 3.4, the following theorem is pertinent. As always, we assume that G is connected and has an infinite node set X.

Theorem 3.7. *Let G be locally finite. Then, ${}^*G$ has at least one hypernode not in its principal galaxy Γ_0 and thus at least one galaxy Γ_1 different from Γ_0.*

Proof. Choose any $x_0 \in X$. By connectedness and local finiteness, for each $n \in \mathbf{N}$, the set X_n of nodes that are at a distance of n from x_0 is nonempty and finite. Also, $\cup X_n = X$ by the connectedness of G. By König's Lemma [4, page 40], there is a one-ended path P starting at x_0. P must pass through every X_n. Thus, there is a subsequence $\langle x_0, x_1, x_2, \ldots \rangle$ of the sequence of nodes of P such that $x_n \in X_n$; that is, $d(x_n, x_0) = n$ for every n. Set $\mathbf{x} = [x_n]$. Then, $\mathbf{x}$ must be in a galaxy Γ_1 that is different from the principal galaxy Γ_0. $\square$

4　When ${}^*G$ Has a Hypernode Not in Its Principal Galaxy

In this section, G is connected and infinite but not necessarily locally finite. Let Γ_a and Γ_b be two galaxies that are different from the principal galaxy Γ_0 of ${}^*G$. We shall say that Γ_a *is closer to* Γ_0 *than is* Γ_b and that Γ_b *is further away from* Γ_0 *than is* Γ_a if there are a $\mathbf{y} = [y_n]$ in Γ_a and a $\mathbf{z} = [z_n]$ in Γ_b such that, for some $\mathbf{x} = [x_n]$ in Γ_0 and for every $m_0 \in \mathbf{N}$, we have

$$N_0(m_0) \;=\; \{n \colon d(z_n, x_n) - d(y_n, x_n) \geq m_0\} \;\in\; \mathcal{F}.$$

Any set of galaxies for which every two of them, say, Γ_a and Γ_b satisfy this condition will be said to be *totally ordered according to their closeness to* Γ_0. With Lemma 3.1 in hand, the conditions for a total ordering (reflexivity, antisymmetry, transitivity, and connectedness) are readily shown. For instance, the proof of Theorem 4.3 below establishes transitivity.

Lemma 4.1. *These definitions are independent of the representative sequences* $\langle x_n \rangle$, $\langle y_n \rangle$, *and* $\langle z_n \rangle$ *chosen for* $\mathbf{x}$, $\mathbf{y}$, *and* $\mathbf{z}$.

Proof. Let $\langle x'_n \rangle$, $\langle y'_n \rangle$, and $\langle z'_n \rangle$ be any other such representative sequences. Then,

$$d(z_n, x_n) \;\leq\; d(z_n, z'_n) + d(z'_n, x'_n) + d(x'_n, x_n).$$

So,

$$d(z'_n, x'_n) \;\geq\; d(z_n, x_n) - d(z_n, z'_n) - d(x'_n, x_n) \;\geq\; d(z_n, x_n) - m_1$$

for some $m_1 \in \mathbf{N}$ and for all n in some $N_1(m_1) \in \mathcal{F}$. Also,

$$d(y'_n, x'_n) \;\leq\; d(y'_n, y_n) + d(y_n, x_n) + d(x_n, x'_n) \;\leq\; d(y_n, x_n) + m_2$$

for some $m_2 \in \mathbf{N}$ and for all n in some $N_2(m_2) \in \mathcal{F}$. Therefore,

$$d(z'_n, x'_n) - d(y'_n, x'_n) \;\geq\; d(z_n, x_n) - d(y_n, x_n) - m_1 - m_2$$

for all n in $N_1(m_1) \cap N_2(m_2) \in \mathcal{F}$. So, for $N_0(m_0)$ as defined above and for each m_0 no matter how large,

$$\{n \colon d(z'_n, x'_n) - d(y'_n, x'_n) \;\geq\; m_0 - m_1 - m_2\} \;\supseteq\; N_0(m_0) \cap N_1(m_1) \cap N_2(m_2) \;\in\; \mathcal{F}.$$

This proves Lemma 4.1. $\square$

We will say that a set A is a *totally ordered, two-way infinite sequence* if there is a bijection from the set $\mathbf{Z}$ of integers to the set A that preserves the total ordering of $\mathbf{Z}$.

Theorem 4.2. *If $^*G$ has a hypernode that is not in its principal galaxy Γ_0, then there exists a two-way infinite sequence of galaxies totally ordered according to their closeness to Γ_0.*

Note. There may be many such sequences, and a galaxy in one sequence and a galaxy in another sequence may not be comparable according to their closeness to Γ_0.

Proof. Let $\mathbf{x} = [\langle x, x, x, \ldots \rangle]$ be a standard hypernode in Γ_0. Also, let $\mathbf{v} = [v_n]$ be the asserted hypernode not in Γ_0. Thus, for each $m \in \mathbf{N}$, $\{n : d(v_n, x) > m\} \in \mathcal{F}$. We can choose a subsequence $\langle y_n \rangle$ of $\langle v_n \rangle$ such that $d(y_n, x))$ is a monotonically increasing sequence of natural numbers that tends to ∞ as $n \to \infty$. Thus, $\mathbf{y} = [y_n]$ is a hypernode in a galaxy Γ_b different from Γ_0.

There will be a smallest $n_1 \in \mathbf{N}$ such that $d(y_n, x) - d(y_0, x) > 1$ for all $n \geq n_1$. Set $w_n = y_0$ for $0 \leq n < n_1$. Thus, for $0 \leq n < n_1$, we have that $d(y_n, x) - d(w_n, x) \geq 0$ and $d(w_n, x) \geq 0$.

Again, there will be a smallest $n_2 \in \mathbf{N}$ such that $d(y_n, x) - d(y_{n_1}, x) > 2$ for all $n \geq n_2$. Set $w_n = y_0$ for $n_1 \leq n < n_2$. Thus, for $n_1 \leq n < n_2$, we have that $d(y_n, x) - d(w_n, x) > 1$ and $d(w_n, x) \geq 0$.

Once again, there will be a smallest $n_3 \in \mathbf{N}$ such that $d(y_n, x) - d(y_{n_2}, x) > 3$ for all $n \geq n_3$. Set $w_n = y_{n_1}$ for $n_2 \leq n < n_3$. Thus, for $n_2 \leq n < n_3$, we have that $d(y_n, x) - d(w_n, x) > 2$ and $d(w_n, x) > 1$. The last inequality follows from $d(y_{n_1}, x) > d(y_0, x) + 1 \geq 1$ for all $n \geq n_1$.

Continuing this way, we will have a smallest $n_k \in \mathbf{N}$ such that $d(y_n, x) - d(y_{n_{k-1}}, x) > k$ for all $n \geq n_k$. Set $w_n = y_{n_{k-2}}$ for $n_{k-1} \leq n < n_k$. In this general case for $n_{k-1} \leq n < n_k$, we have that $d(y_n, x) - d(w_n, x) > k - 1$ and $d(w_n, x) > k - 2$. The last inequality occurs because $d(y_{n_{k-2}}, x) > d(y_{n_{k-3}}, x) + k - 2 > k - 2$ for all $n \geq n_{k-2}$.

Altogether then, w_n is defined for all n. Moreover, $d(w_n, x)$ increases monotonically, eventually becoming larger than m for every $m \in \mathbf{N}$. Therefore, $\mathbf{w} = [w_n]$ is in a galaxy Γ_a different from the principal galaxy Γ_0. Furthermore, $d(y_n, x) - d(w_n, x)$ also increases

monotonically in the same way. Consequently, the galaxy Γ_a containing $\mathbf{w} = [w_n]$ is closer to Γ_0 than is the galaxy Γ_b containing $\mathbf{y} = [y_n]$.

We can now repeat this argument with Γ_b replaced by Γ_a and with $\mathbf{w} = [w_n]$ playing the role that $\mathbf{y} = [y_n]$ played to find still another galaxy Γ'_a different from Γ_0 and closer to Γ_0 than is Γ_a. Continual repetitions yield an infinite sequence of galaxies indexed by, say, the negative integers and totally ordered by their closeness to Γ_0.

The conclusion that there is an infinite sequence of galaxies progressively further away from Γ_0 than is Γ_b is easier to prove. With $\mathbf{y} \in \Gamma_b$ as before, we have that, for every $m \in \mathbb{N}$, $\{n : d(y_n, x) > m\} \in \mathcal{F}$. Therefore, for each $n \in \mathbb{N}$, we can choose z_n as an element of $\langle y_n \rangle$ such that $d(z_n, x) \geq d(y_n, x) + n$ and also such that $d(z_n, x)$ monotonically increases with n. Clearly, $d(z_n, x) \to \infty$ as $n \to \infty$. This implies that $\mathbf{z} = [z_n]$ must be in a galaxy Γ_c that is further away from Γ_0 than is Γ_b

We can repeat the argument of the last paragraph with Γ_c in place of Γ_b to find still another galaxy Γ'_c further away from Γ_0 than is Γ_c. Repetitions of this argument show that there is an infinite sequence of galaxies indexed by, say, the positive integers and totally ordered by their closeness to Γ_0. The union of the two infinite sequences yields the conclusion of the theorem. $\square$

By virtue of Theorem 3.7, the conclusion of Theorem 4.2 holds whenever G is locally finite.

In general, the hypothesis of Theorem 4.2 may or may not hold. Thus, $^*G$ either has exactly one galaxy, its principal one Γ_0, or has infinitely many galaxies.

A more general result may be true: Namely, for every two galaxies Γ_1 and Γ_3 different from Γ_0 with Γ_1 closer to Γ_0 than is Γ_3, there is another galaxy Γ_2 with Γ_2 further away from (resp. closer to) Γ_0 than is Γ_1 (resp. Γ_3). This has yet to be proven.

Instead of the idea of "totally ordered according to closeness to Γ_0," we can define the idea of "partially ordered according to closeness to Γ_0" in much the same way. Just drop the connectedness axiom for a total ordering.

Theorem 4.3. *Under the hypothesis of Theorem 4.2, the set of galaxies of $^*G$ is partially ordered according to the closeness of the galaxies to the principal galaxy Γ_0.*

Proof. Reflexivity and antisymmetry are obvious. Consider transitivity: Let Γ_a, Γ_b, and Γ_c be galaxies different from Γ_0. (The case where $\Gamma_a = \Gamma_0$ can be argued similarly.) Assume that Γ_a is closer to Γ_0 than is Γ_b and that Γ_b is closer to Γ_0 than is Γ_c. Thus, for any $\mathbf{x}$ in Γ_0, $\mathbf{u}$ in Γ_a, $\mathbf{v}$ in Γ_b, and $\mathbf{w}$ in Γ_c and for every $m \in N$, we have

$$N_{uv} = \{n \colon d(v_n, x_n) - d(u_n, x_n) \geq m\} \in \mathcal{F}$$

and

$$N_{vw} = \{n \colon d(w_n, x_n) - d(v_n, x_n) \geq m\} \in \mathcal{F}.$$

We also have

$$d(w_n, x_n) - d(u_n, x_n) = d(w_n, x_n) - d(v_n, x_n) + d(v_n, x_n) - d(u_n, x_n).$$

So,

$$N_{uw} = \{n \colon d(w_n, x_n) - d(u_n, x_n) \geq 2m\} \supseteq N_{uv} \cap N_{vw} \in \mathcal{F}.$$

Thus, $N_{uw} \in \mathcal{F}$. Since m can be chosen arbitrarily, we can conclude that Γ_a is closer to Γ_0 than is Γ_c. $\square$

5   The Hyperordinals

In the following sections, we shall extend the results obtained so far to enlargements of transfinite graphs of rank 1, that is, to enlargements of 1-graphs. For this purpose, we need to replace the set $^*N$ of hypernaturals by a set of "hyperordinals"; these are defined as follows. A hyperordinal $\underline{\alpha}$ is an equivalence class of sequences of ordinals where two such sequences $\langle \alpha_n \rangle$ and $\langle \beta_n \rangle$ are taken to be equivalent if $\{n \colon \alpha_n = \beta_n\} \in \mathcal{F}$. We denote $\underline{\alpha}$ also by $[\alpha_n]$ where again the α_n are the elements of one (any one) of the sequences in the equivalent class. Any set of hyperordinals is totally ordered by the inequality relation. That is, given any hyperordinals $\underline{\alpha} = [\alpha_n]$ and $\underline{\beta} = [\beta_n]$, exactly one of the sets:

$$\{n \colon \alpha_n < \beta_n\}, \quad \{n \colon \alpha_n = \beta_n\}, \quad \{n \colon \alpha_n > \beta_n\}$$

will be in $\mathcal{F}$. So, exactly one of the expressions:

$$\underline{\alpha} < \underline{\beta}, \quad \underline{\alpha} = \underline{\beta}, \quad \underline{\alpha} > \underline{\beta}$$

holds.

6   Walks in 1-Graphs

1-graphs arise when conventionally infinite graphs are connected at their infinite extremities through 1-nodes, the latter being a generalization of the idea of a node. Such 1-nodes and the resulting 1-graphs are defined in [5, Section 2.1] and also in [6, Section 2.3]. Let us restate the needed definitions concisely.

We will be dealing with two kinds of nodes and two kinds of graphs. A conventionally infinite graph G^0 will now be called a 0-*graph* and the nodes in G^0 will be called 0-*nodes* in order to distinguish these ideas from those pertaining to transfinite graphs of rank 1. Similarly, what we called a "hypernode" previously will henceforth be called a 0-*hypernode*, and what we called a "galaxy" in the enlargement of a 0-graph will now be called a 0-*galaxy*.

An *infinite extremity* of a 0-graph G^0 is defined as an equivalence class of one-ended paths in G^0, where two such paths are considered to be *equivalent* if they are eventually identical. Such an equivalence class is called a 0-*tip* of G^0. G^0 may have one or more 0-tips (or possibly none at all). To obtain the "1-nodes," the set of 0-tips is partitioned in some fashion into subsets, and to each subset a single 0-node may (or may not) be added under the proviso that, if a 0-node is added to one subset, it is not added to any other subset. Then, each subset (possibly augmented with a 0-node) is called a 1-*node*. With X^1 denoting the set of 1-nodes and X^0 the set of 0-nodes of G^0, the 1-*graph* G^1 is defined as the triplet:

$$G^1 \;=\; \{X^0, B, X^1\},$$

and $G^0 = \{X^0, B\}$ is now called the 0-*graph of* G^1. Furthermore, a path in G^0 is now called a 0-*path*, and connectedness in G^0 is now called 0-*connectedness*. We will consistently append the superscript 0 to the symbols and the prefix 0- to the terminology for concepts from Sections 2 through 4 regarding 0-graphs.

In order to define the "1-galaxies," we need the idea of distances in a 1-graph G^1. But now, we must make a significant choice. The distances between two nodes (0-nodes or 1-nodes) can be defined as the minimum length of all paths—or, alternatively, of all walks—connecting the two nodes. It turns out that a path need not exist between two nodes in a 1-graph G^1, but a walk always will exist between them. To ensure the existence

of at least one path between every two nodes, additional conditions must be imposed on G^1 (see [5, Conditions 3.2-1 and 3.5-1] or [6, Condition 3.1-2]), and this leads to a more restrictive and yet more complicated theory involving distances. Such can be done, but it is more general and simpler to use walk-based distance ideas. This we now do.

A *nontrivial 0-walk* W^0 in a 0-graph is the conventional concept. It is a (finite or one-way infinite or two-way infinite) alternating sequence:

$$W^0 \; = \; \langle \ldots, x^0_{-1}, b_{-1}, x^0_0, b_0, x^0_1, b_1, \ldots \rangle \tag{2}$$

of 0-nodes x^0_m and branches b_m, where each branch b_m is incident to the two 0-nodes x^0_m and x^0_{m+1} adjacent to it in the sequence. If the sequence terminates at either side, it is required to terminate at a 0-node. The 0-walk is called *two-ended* or *finite* if it terminates on both sides, *one-ended* if it terminates on just one side, and *endless* if it terminates on neither side.

A *trivial 0-walk* is a singleton set whose sole element is a 0-node.

A one-ended 0-walk W^0 will be called *extended* if its 0-nodes are eventually distinct, that is, if it is eventually identical to a one-ended path. We say that W^0 *traverses* a 0-tip if it is extended and eventually identical to a representative of that 0-tip. Finally, W^0 is said to *reach* a 1-node x^1 if W^0 traverses a 0-tip contained in x^1. In the same way, an endless 0-walk can *reach* two 1-nodes (or possibly reach the same 1-node) by traversing two 0-tips, one toward the left and the other toward the right. When this is so, we say that the endless 0-walk is *extended*. On the other hand, if a 0-walk terminates at a 0-node contained in a 1-node, we again say that the 0-walk *reaches* both of those nodes and does so *through* a branch incident to that 0-node.

Every two-ended 0-walk contains a 0-path that terminates at the two 0-nodes at which the 0-walk terminates, so there is no need to employ 0-walks when defining distances in a 0-graph. On the other hand, such a need arises for 1-graphs. To meet this need, we first define a 0-*section* S^0 in a 1-graph G^1 as a subgraph S^0 of the 0-graph G^0 of G^1 induced by a maximal set of branches that are pairwise 0-connected in G^0. A 1-node x^1 is said to be *incident to* S^0 if either it contains a 0-node incident to a branch of S^0 or it contains a 0-tip having a representative one-ended path lying entirely within S^0. In this case, we also say

that that 0-tip *belongs to* S^0. Given two 1-nodes x^1 and y^1 incident to S^0, there will be a 0-walk W^0 in S^0 that reaches each of x^1 and y^1 through a 0-tip belonging to S^0 or through a branch in S^0.[4] Moreover, there may also be a 0-walk W^0 in S^0 that reaches the same 1-node at both extremities of W^0. To be more specific, let us state

Lemma 6.1. *Let S^0 be a 0-section in G^1, and let x^1 and y^1 be two 1-nodes incident to S^0. Then, there exists a 0-walk in S^0 that reaches x^1 and y^1.*

Proof. That x^1 is incident to S^0 means that there is a 0-path P_x^0 in S^0 that either reaches x^1 through a 0-tip of x^1 or reaches x^1 through a branch. Similarly, there is such a 0-path P_y^0 reaching y^1. Let u^0 be a node of P_x^0, and let v^0 be a node of P_y^0. Since S^0 is 0-connected, there is a 0-path P_{uv}^0 in S^0 terminating at u^0 and v^0 (possibly a trivial 0-path if $u^0 = v^0$). Then, $P_x^0 \cup P_{uv}^0 \cup P_y^0$ as a 0-walk in S^0 as asserted. □

A *nontrivial, two-ended 1-walk* W^1 is a finite sequence:

$$W^1 \;=\; \langle x_0, W_0^0, x_1^1, W_1^0, \ldots, x_{m-1}^1, W_{m-1}^0, x_m \rangle \tag{3}$$

with $m \geq 1$ that satisfies the following conditions.

1. $x_1^1, \ldots, x_{m-1}^1$ are 1-nodes, while x_0 and x_m may be either 0-nodes or 1-nodes.

2. For each $k = 0, \ldots, m-1$, W_k^0 is a nontrivial 0-walk that reaches the two nodes adjacent to it in the sequence.

3. For each $k = 1, \ldots, m-1$, at least one of W_{k-1}^0 and W_k^0 reaches x_k^1 through a 0-tip, not through a branch.

A *one-ended* 1-walk is a sequence like (3) except that it extends infinitely to the right. An *endless* 1-walk extends infinitely on both sides. A *trivial 1-walk* is a singleton set whose sole element is either a 0-node or a 1-node.

We now define a more general kind of connectedness (called "1-wconnectedness" to distinguish it from path-based 1-connectedness). Two branches (resp. two nodes—either 0-nodes or 1-nodes) will be said to be 1-*wconnected* if there exists a 0-walk or 1-walk that

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[4] For examples of when a 0-walk is needed because a 0-path won't do, see Figures 3.1 and 3.2 of [5] and Figures 4.1, 5.1, 5.2, and 5.3 of [6].

terminates at a 0-node of each branch (resp. that terminates at those two nodes). If a terminal node of a walk is the same as, or contains, or is contained in the terminal node of another walk, the two walks taken together form another walk. We call this the *conjunction* of the two walks. It follows that 1-wconnectedness is a transitive binary relation for the branch set B of the 1-graph G^1 and is in fact an equivalence relation. If every two branches of G^1 are 1-wconnected, we will say that G^1 is 1-*wconnected*.

7 Walk-Based Distances in a 1-Graph

The length $|W^0|$ of a 0-walk W^0 is defined as follows: If W^0 is two-ended, $|W^0|$ is the number τ_0 of branch traversals in it; that is, each branch is counted as many times as it appears in W^0. If W^0 is one-ended and extended, we set $|W^0| = \omega$, the first transfinite ordinal. If W^0 is endless and extended in both directions, we set $|W^0| = \omega \cdot 2$.

As for a nontrivial two-ended 1-walk W^1, its length $|W^1|$ is taken to be $|W^1| = \sum_{k=0}^{m} |W_k^0|$, where the sum is over the finitely many 0-walks W_k^0 in (3). Thus,

$$|W^1| \; = \; \omega \cdot \tau_1 + \tau_0 \tag{4}$$

where τ_1 is the number of traversals of 0-tips performed by W^1 and τ_0 is the number of traversals of branches in all the two-ended (i.e., finite) 0-walks appearing as terms in (3). We take $\sum_{k=0}^{m} |W_k^0|$ to be the natural sum of ordinals; this yields a normal expansion of an ordinal [1, pages 354-355]. τ_1 is not 0 because W^1 is a nontrivial, two-sided 1-walk. However, τ_0 may be 0, this occurring when every W_k^0 in (3) is one-ended or endless.

A 0-node is called *maximal* if it is not contained in a 1-node, and *nonmaximal* otherwise. A distance measured from a nonmaximal 0-node is the same as that measured from the 1-node containing it. Given two nodes x and y (of ranks 0 or 1), we define the *wdistance*[5] $d(x, y)$ between them as

$$d(x, y) \; = \; \min |W_{x,y}| \tag{5}$$

where the minimum is taken over all two-ended walks (0-walks or 1-walks) terminating at x and y. That minimum exists because any set of ordinals is a well-ordered set. In view of (4), $d(x, y) < \omega^2$. If $x = y$, we set $d(x, x) = 0$.

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[5]We write "wdistance" to distinguish this walk-based idea from a distance based on paths.

Clearly, if $x \neq y$, $d(x, y) > 0$ and $d(x, y) = d(y, x)$. Furthermore, the conjunction of two two-ended walks is again a two-ended walk, whose length is the natural sum of the ordinal lengths of the two walks. So, by taking minimums appropriately, we obtain the triangle inequality:

$$d(x, z) \ \leq \ d(x, y) \ + \ d(y, z) \tag{6}$$

where again the natural sum of ordinals is understood. Altogether then, we have

Lemma 7.1. *The ordinal-valued wdistances between the maximal nodes of a 1-graph satisfy the metric axioms.*

8 Enlargements of 1-Graphs and Hyperdistances in Them

In [6, pages 163-164], a nonstandard 1-node was defined as an equivalence class of sequences of sets of tips shorted together, with the tips taken from sequences of possibly differing 1-graphs. But, since each set of tips shorted together is a 1-node, that definition of a nonstandard 1-node can also be stated as an equivalence class of sequences of 1-nodes. Specializing to the case where all the 1-graphs are the same, we have the following definition of a nonstandard 1-node, which we now call a "1-hypernode."

Consider a given 1-graph along with a chosen free ultrafilter $\mathcal{F}$. Two sequences $\langle x_n^1 \rangle$ and $\langle y_n^1 \rangle$ of 1-nodes in G^1 are taken to be *equivalent* if $\{n : x_n^1 = y_n^1\} \in \mathcal{F}$. It is easy to show that this is truly an equivalence relation. Then, $\mathbf{x}^1 = [x_n^1]$ denotes one such equivalence class, where the x_n^1 are the elements of any one of the sequences in that class. $\mathbf{x}^1$ will be called a 1-*hypernode*.

The *enlargement* of the 1-graph $G^1 = \{X^0, B, X^1\}$ is the nonstandard 1-graph

$$^*G^1 \ = \ \{\,^*X^0, \,^*B, \,^*X^1\,\}$$

where $^*X^0$ and $^*B$ are respectively the set of 0-hypernodes and branches in the enlargement of the 0-graph $G^0 = \{X^0, B\}$ of G^1 and $^*X^1$ is the set of 1-hypernodes defined above, that is, the set of all equivalence classes of sequences of 1-nodes taken from X^1.

We define the *hyperdistance* $\mathbf{d}$ between any two hypernodes $\mathbf{x}$ and $\mathbf{y}$ of $^*G^1$ (of ranks 0

and/or 1) to be the internal function

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) \;=\; [d(x_n, y_n)]. \tag{7}$$

Since distances in G^1 are less than ω^2, $\mathbf{d}(\mathbf{x}, \mathbf{y})$ is a hyperordinal less than $\underline{\omega}^2$. We say that a 0-hypernode $\mathbf{x}^0 = [x_n^0]$ is *maximal* if the set of n for which x_n^0 is not contained in a 1-node is a member of $\mathcal{F}$. All the 1-nodes in this work are perforce maximal because there are no nodes of higher rank. $\mathbf{d}$, when restricted to the maximal hypernodes, also satisfies the metric axioms, in particular, the triangle inequality:

$$\mathbf{d}(\mathbf{x}, \mathbf{z}) \;\leq\; \mathbf{d}(\mathbf{x}, \mathbf{y}) + \mathbf{d}(\mathbf{y}, \mathbf{z}) \tag{8}$$

But, now $\mathbf{d}$ is hyperordinal-valued.

9 The Galaxies of ${}^*G^1$

The 0-*galaxies* of ${}^*G^1$ are defined just as they are for the enlargement ${}^*G^0$ of a 0-graph; see Section 3. However, we henceforth write "0-galaxy" in place of "galaxy" and "0-limitedly distant" in place of "limitedly distant."

As was mentioned above, each 0-section of G^1 is the subgraph of the 0-graph $G^0\{X^0, B\}$ of G^1 induced by a maximal set of branches that are 0-connected. A 0-section is a 0-graph by itself. So, within the enlargement ${}^*G^1$, each 0-section S^0 enlarges into ${}^*S^0$ as defined in Section 2. Within each enlarged 0-section there may be one or more 0-galaxies. As a special case, a particular 0-section may have only finitely many 0-nodes, and so its enlargement is itself—all its 0-hypernodes are standard. On the other hand, there may be infinitely many 0-galaxies in some enlarged 0-section. Moreover, the enlarged 0-sections do not, in general, comprise all of the enlarged 0-graph ${}^*G^0 = \{\, {}^*X^0, \, {}^*B \,\}$ of ${}^*G^1$. Indeed, there can be a 0-hypernode $\mathbf{x}^0 = [x_n^0]$ where each x_n^0 resides in a different 0-section; in this case $\mathbf{x}^0$ will reside in a 0-galaxy that is not in an enlargement of a 0-section.

Something more can happen with regard to the 0-galaxies in ${}^*G^1$. 0-galaxies can now contain 1-hypernodes. For example, this occurs when a 1-node x^1 is incident to a 0-section S^0 through a branch. Then, the standard 1-hypernode $\mathbf{x}^1$ corresponding to x^1 is 0-limitedly distant from the standard 0-hypernodes in ${}^*S^0$. So, there is a 0-galaxy containing not only

$^*S^0$ but $\mathbf{x}^1$ as well. See Example 9.3 below in this regard. In general, the nodal 0-galaxies partition the set $^*X^0 \cup {}^*X^1$ of all the hypernodes in $^*G^1$. As we shall see in Examples 9.1 and 9.2 below, there may be a singleton 0-galaxy containing a 1-hypernode only.

Let us now turn to the "1-galaxies" of $^*G^1$ Two hypernodes $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ (of ranks 0 and/or 1) in $^*G^1$ will be said to be in the same *nodal 1-galaxy* $\dot{\Gamma}^1$ if there exists a natural number $k \in \mathbf{N}$ such that $\{n: d(x_n, y_n) \leq \omega \cdot k\} \in \mathcal{F}$. In this case, we say that $\mathbf{x}$ and $\mathbf{y}$ are *1-limitedly distant*, and we write $\mathbf{d}(\mathbf{x}, \mathbf{y}) \leq [\omega \cdot k]$ where $[\omega \cdot k]$ denotes the standard hyperordinal corresponding to $\omega \cdot k$. This defines an equivalence relation on the set $^*X^0 \cup {}^*X^1$ of all the hypernodes in $^*G^1$. Indeed, reflexivity and symmetry are obvious. For transitivity, assume that $\mathbf{x}$ and $\mathbf{y}$ are 1-limitedly distant and that $\mathbf{y}$ and $\mathbf{z}$ are 1-limitedly distant, too. Then, there are natural numbers k_1 and k_2 such that

$$N_{xy} = \{n: d(x_n, y_n) \leq \omega \cdot k_1\} \in \mathcal{F}$$

and

$$N_{yz} = \{n: d(y_n, z_n) \leq \omega \cdot k_2\} \in \mathcal{F}.$$

By the triangle inequality (6),

$$N_{xz} = \{n: d(x_n, z_n) \leq \omega \cdot (k_1 + k_2)\} \supseteq N_{xy} \cap N_{yz} \in \mathcal{F}.$$

So, $N_{xz} \in \mathcal{F}$ and therefore $\mathbf{x}$ and $\mathbf{z}$ are 1-limitedly distant. We can conclude that the set $^*X^0 \cup {}^*X^1$ of all hypernodes in $^*G^1$ is partitioned into nodal 1-galaxies by this equivalence relation.

Corresponding to each nodal 1-galaxy $\dot{\Gamma}^1$, we define a 1-*galaxy* Γ^1 as a nonstandard subgraph of $^*G^1$ consisting of all the hypernodes in $\dot{\Gamma}^1$ along with all the hyperbranches both of whose 0-hypernodes are in $\dot{\Gamma}^1$.

No hyperbranch can have its two incident 0-hypernodes in two different 0-galaxies or two different 1-galaxies because the distance between their 0-hypernodes is 1. Thus, the hyperbranch set $^*B$ is also partitioned by the 0-galaxies and more coarsely by the 1-galaxies.

The *principal 1-galaxy* Γ_0^1 of $^*G^1$ is the 1-galaxy whose hypernodes are 1-limitedly distant from a standard hypernode in $^*G^1$ (i.e., from a node of G^1).

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Note that the enlargement $^*S^0$ of each 0-section S^0 of G^1 has its own principal 0-galaxy $\Gamma_0^0(S^0)$. Moreover, every $^*S^0$ lies within the principal 1-galaxy Γ_0^1. Indeed, any standard hypernode $\mathbf{x}$ by which Γ_0^1 may be defined and any standard 0-hypernode $\mathbf{y}^0$ by which $\Gamma_0^0(S^0)$ may be defined are 1-limitedly distant. Also, the hyperdistance $\mathbf{d}(\mathbf{y}^0, \mathbf{z}^0)$ between any two 0-hypernodes $\mathbf{y}^0$ and $\mathbf{z}^0$ of $^*S^0$ is no larger than a hypernatural $\mathbf{k}$. So, by the triangle inequality (8), every 0-hypernode of $^*S$ is 1-limitedly distant from $\mathbf{x}$. Whence our assertion.

Example 9.1. Consider an endless 1-path P^1 having an endless 0-path between every consecutive pair of 1-nodes in P^1. The 0-sections of P^1 are those endless 0-paths, and each of their enlargements have an infinity of 0-galaxies in $^*P^1$. However, there are other 0-galaxies in $^*P^1$, infinitely many of them. Indeed, consider a 0-hypernode $\mathbf{x}^0 = [x_n^0]$, where each 0-node x_n^0 lies in a different 0-section of P^1; $\mathbf{x}^0$ will lie in a 0-galaxy Γ_1^0 different from all the 0-galaxies in any enlargement of a 0-section of P^1. The 0-nodes of Γ_1^0 will be all the 0-hypernodes that are 0-limitedly distant from $\mathbf{x}^0$. Furthermore, there are still other 0-galaxies now. Each 1-hypernode $\mathbf{x}^1 = [x_n^1]$ is the sole member of a 0-galaxy. In fact, the nodal 0-galaxies partition the set of all the 0-hypernodes and 1-hypernodes.

On the other hand, the principal 1-galaxy of $^*P^1$ consists of all the standard 1-hypernodes corresponding to the 1-nodes of P^1 along with the enlargements of the 0-sections of P^1. Also, there will be infinitely many 1-galaxies, each of which contains infinitely many 0-galaxies along with 1-hypernodes. In this particular case, each of the 1-galaxies is graphically isomorphic to the principal 1-galaxy, but this is not true in general. □

Example 9.2. An example of a nonstandard 1-graph $^*G^1$ having exactly one 1-galaxy (its principal one) and infinitely many 0-galaxies is provided by the enlargement of the 1-graph G^1 obtained from the 0-graph of Figure 1 by replacing each branch by an endless 0-path, thereby converting each 0-node into a 1-node. Again each endless path of that 1-graph G^1 is a 0-section, and its enlargement is like that of Example 3.2. There are infinitely many such 0-galaxies in the enlargement $^*G^1$ of G^1. Also, there are infinitely many 0-galaxies, each consisting of a single 1-hypernode. With regard to the 1-galaxies, the enlargement $^*G^1$ of G^1 mimics that of Example 3.4, except that now the rank 0 is replaced by the rank 1. The hyperdistance between every two 1-hypernodes (resp. 0-hypernodes) is no larger than

$\omega \cdot 4$ (resp. $\omega \cdot 6$). Hence, $^*G^1$ has only one 1-galaxy, its principal one. □

Example 9.3. Here is an example where the 1-hypernodes are not isolated within 0-galaxies. Replace each of the horizontal branches in Figure 1 by an endless 0-path, but do not alter the branches incident to x_g. Now, the nodes x_k ($k = 0, 1, 2, \ldots$) become 1-nodes x_k^1, each containing a 0-node of the branch incident to x_k^1 and x_g. The corresponding standard 1-hypernodes along with the standard 0-hypernode for x_g and the standard hyperbranches connecting them all comprise a single 0-galaxy. Moreover, there will be other 0-galaxies obtained through equivalence classes of sequences of these nodes and branches. The endless paths that replace the horizontal branches lead to still other 0-galaxies. Again, the nodal 0-galaxies partition the set of all the hypernodes in $^*G^1$.

On the other hand, there is again only one 1-galaxy for $^*G^1$. □

Example 9.4. The distances in the three preceding examples can be fully defined by paths. So, let us now present an example where walks are needed. The 1-graph G^1 of Figure 3 illustrates one such case. It consists of an infinite sequence of 0-subgraphs, each of which is an infinite series connections of four-branch subgraphs, each in a diamond configuration, as shown. To save words, we shall refer to such an infinite series connection as a "chain." The chain starting at the 0-node x_k^0 will be denoted by C_k ($k = 0, 1, 2, \ldots$). Each C_k is a 0-graph; it does not contain any 1-node. Each C_k has uncountably many 0-tips. One 0-tip has a representative 0-path starting at x_k^0, proceeding along the left-hand sides of the diamond configurations, and reaching the 1-node x_k^1. Another 0-tip has a representative 0-path that proceeds along the right-hand sides and reaches the 1-node x_{k+1}^1. Still other 0-tips of C_k (uncountably many of them) have representatives that pass back and forth between the two sides infinitely often to reach singleton 1-nodes; these are not shown in that figure. The chain C_k is connected to C_{k+1} through the 1-node x_{k+1}^1, as shown. Note that there is no path connecting, say, x_k^0 to x_m^0 when $m - k \geq 2$, but there is such a walk.

Each C_k is a 0-section, and its enlargement $^*C_k$ has infinitely many 0-galaxies. Also, the 1-nodes x_k^1 together produce infinitely many 0-galaxies, each being a single 1-hypernode. As before, the nodal 0-galaxies comprise a partition of $^*X^0 \cup {}^*X^1$.

On the other hand, the enlargement $^*G^1$ of the 1-graph G^1 of Figure 3 has infinitely

many 1- galaxies. Its principal one is a copy of G^1. Each of the other 1-galaxies is also a copy of G^1 except that it extends infinitely in both directions—infinitely to the left and infinitely to the right. Here, too, the nodal 1-galaxies comprise a partitioning of ${}^*X^0 \cup {}^*X^1$, but a coarser one. □

These examples indicate that the enlargements of 1-graphs can have rather complicated structures.

10 Locally 1-Finite 1-Graphs and a Property of Their Enlargements

In general, ${}^*G^1$ has 1-galaxies other than its principal 1-galaxy. One circumstance where this occurs is when ${}^*G^1$ is locally finite in certain way, which we will explicate below.

We need some more definitions. Two 1-nodes of G^1 are said to be 1-*adjacent* if they are incident to the same 0-section. A 1-node will be called a *boundary 1-node* if it is incident to two or more 0-sections. G^1 will be called *locally 1-finite* if each of its 0-sections has only finitely many incident boundary 1-nodes.[6]

Lemma 10.1. *Let x^1 be a boundary 1-node. Then, any 1-walk that passes through x^1 from any 0-section S_1^0 incident to x^1 to any other 0-section S_2^0 incident to x^1 must have a length no less than ω.*

Proof. The only way such a walk can have a length less than ω (i.e., a length equal to a natural number) is if it avoids traversing a 0-tip in x^1. But, this means that it passes through two branches incident to a 0-node in x^1. But, that in turn means that S_1^0 and S_2^0 cannot be different 0-sections. □

Remember that G^1 is called 1-wconnected if, for every two nodes of G^1, there is a 0-walk or 1-walk that reaches those two nodes.

Lemma 10.2. *Any two 1-nodes x^1 and y^1 that are 1-wconnected but are not 1-wadjacent must satisfy $d(x^1, y^1) \geq \omega$.*

Proof. Any walk 1-wconnecting x^1 and y^1 must pass through at least one boundary 1-node different x^1 and y^1 while passing from one 0-section to another 0-section. Therefore,

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[6]Note that a 0-section in a locally 1-finite 1-graph may have infinitely many incident 1-nodes that are not boundary 1-nodes. Also, this definition of locally 1-finiteness does not prohibit 0-nodes of infinite degree.

that walk must be a 1-walk. By Lemma 10.1, its length is no less than ω. Since this is true for every such walk, our conclusion follows. $\square$

The next theorem mimics Theorem 3.7 but at the rank 1.

Theorem 10.3. *Let G^1 be locally 1-finite and 1-wconnected and have infinitely many boundary 1-nodes. Then, given any 1-node x_0^1 of G^1, there is a one-ended 1-walk W^1 starting at x_0^1:*

$$W^1 \;=\; \langle x_0^1, W_0^0, x_1^1, W_1^0, \ldots, x_m^1, W_m^0, \ldots \rangle$$

such that there is a subsequence of 1-nodes $x_{m_k}^1$, $k = 1, 2, 3, \ldots$, satisfying $d(x_0^1, x_{m_k}^1) \geq \omega \cdot k$.

Proof. x_0^1 need not be a boundary 1-node, but it will be 1-wadjacent to only finitely many boundary 1-nodes because of local 1-finiteness and 1-wconnectedness. Let X_0 be the nonempty finite set of those boundary 1-nodes. For the same reasons, there is a nonempty finite set X_1 of boundary 1-nodes, each being 1-wadjacent to some 1-node in X_0 but not 1-wadjacent to x_0^1. By Lemma 10.2, for each $x^1 \in X_2$, we have $d(x_0^1, x^1) \geq \omega$. In general, for each $k \in \mathbb{N}$, $k \geq 2$, there is a nonempty finite set X_k of boundary 1-nodes, each being 1-wadjacent to some 1-node in X_{k-1} but not 1-wadjacent to any of the 1-nodes in $\cup_{l=0}^{k-2} X_l$. By Lemma 10.2 again, for any such $x^1 \in X_k$, we have $d(x_0^1, x^1) \geq \omega \cdot k$.

We now adapt the proof of König's lemma: From each of the infinitely any boundary 1-nodes in G^1, there is a 1-walk reaching that boundary 1-node and also reaching x_0^1. Thus, there are infinitely many 1-walks starting at x_0^1 and passing through one of the 1-nodes in X_0, say, $x_{m_0}^1$. Among those 1-walks, there are again infinitely many 1-walks passing through one of the 1-nodes in X_1, say, $x_{m_1}^1$. Continuing in this say, we find an infinite sequence $\langle x_{m_1}^1, x_{m_2}^1, x_{m_3}^1, \ldots \rangle$ of 1-nodes occurring in a one-ended 1-walk starting at x_0^1 and such that $d(x_0^1, x_{m_k}^1) \geq \omega \cdot k$. $\square$

Corollary 10.4. *Under the hypothesis of Theorem 10.3, the enlargement ${}^*G^1$ of G^1 has at least one 1-hypernode not in its principal galaxy Γ_0^1 and thus at least one 1-galaxy Γ^1 different from its principal 1-galaxy Γ_0^1.*

Proof. Set $\mathbf{x}^1 = [\langle x_0^1, x_{m_0}^1, x_{m_1}^1, \ldots \rangle]$, where the $x_{m_k}^1$ are the 1-nodes specified in the preceding proof. With $\mathbf{x}_0^1$ being the standard 1-hypernode corresponding to x_0^1, we have by Theorem 10.3 that $\mathbf{d}(\mathbf{x}_0^1, \mathbf{x}^1) \geq [\omega \cdot n]$. Hence, $\mathbf{x}^1$ is not 1-limitedly distant from $\mathbf{x}_0^1$ and thus

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must reside in a 1-galaxy Γ^1 different from Γ_0^1. □

11 When ${}^*G^1$ Has a 1-Hypernode Not in Its Principal Galaxy

We are at last ready to extend the results of Section 4 to the rank 1 of transfiniteness. The arguments are the much same as those of Section 4, and so we shall now simply state definitions and results while at times indicating what modifications are needed.

In this section G^1 is 1-wconnected and has an infinity of boundary 1-nodes, but G^1 need not be locally finite. Let Γ_a^1 and Γ_b^1 be two 1-galaxies of ${}^*G^1$ that are different from the principal 1-galaxy Γ_0^1. We say that Γ_a^1 *is closer to* Γ_0^1 *than is* Γ_b^1 and that Γ_b^1 *is further away from* Γ_0^1 *than is* Γ_a^1 if there are a $\mathbf{y} = [y_n]$ in Γ_a^1 and a $\mathbf{z} = [z_n]$ in Γ_b^1 such that, for some $\mathbf{x} = [x_n]$ in Γ_0^1 and for every $m \in \mathbb{N}$,

$$\{n \colon d(z_n, x_n) - d(y_n, x_n) \ \geq \omega \cdot m\} \ \in \ \mathcal{F}.$$

(The ranks of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ may now be either 0 or 1.)

Any set of 1-galaxies for which every two of them, say, Γ_a^1 and Γ_b^1 satisfy these conditions will be said to be *totally ordered according to their closeness to* Γ_0^1. Here, too, the conditions for a total ordering are readily shown.

Lemma 11.1. *These definitions are independent of the representative sequences* $\langle x_n \rangle$, $\langle y_n \rangle$, *and* $\langle z_n \rangle$ *chosen for* $\mathbf{x}$, $\mathbf{y}$, *and* $\mathbf{z}$.

The proof of this lemma is the same as that of Lemma 4.1 except that the rank 0 is replaced by the transfinite rank 1. For instance, the natural numbers m_0, m_1, and m_2 are now replaced by $\omega \cdot m_0$, $\omega \cdot m_1$, and $\omega \cdot m_2$.

Theorem 11.2. *If* ${}^*G^1$ *has a hypernode (of either rank 0 or rank 1) that is not in its principal 1-galaxy* Γ_0^1, *then there exists a two-way infinite sequence of 1-galaxies totally ordered according to their closeness to* Γ_0^1.

Here, too, the proof of this is much like that of Theorem 4.2. For instance, the natural number k is replaced by the ordinal $\omega \cdot k$. Also, galaxies, that is, 0-galaxies are replaced by 1-galaxies.

Similarly, by mimicking the proof of Theorem 4.3, we can prove

Theorem 11.3. *Under the hypothesis of Theorem 11.2, the set of 1-galaxies of $^*G^1$ is partially ordered according to the closeness of the 1-galaxies to Γ_0^1.*

12 Extensions to Higher Ranks of Transfiniteness

The extension of these results to the enlargements of transfinite graphs of any natural-number rank is quite similar to what we have presented. The ideas are the same, but the notations and the details of the arguments are somewhat more complicated. Moreover, further complications arise with the extension to the arrow rank $\vec{\omega}$ of transfiniteness. Extensions to still higher ranks then proceed in much the same way. All this is explicated in the technical report [7], which can also be found in the internet archive www.arxiv.org.

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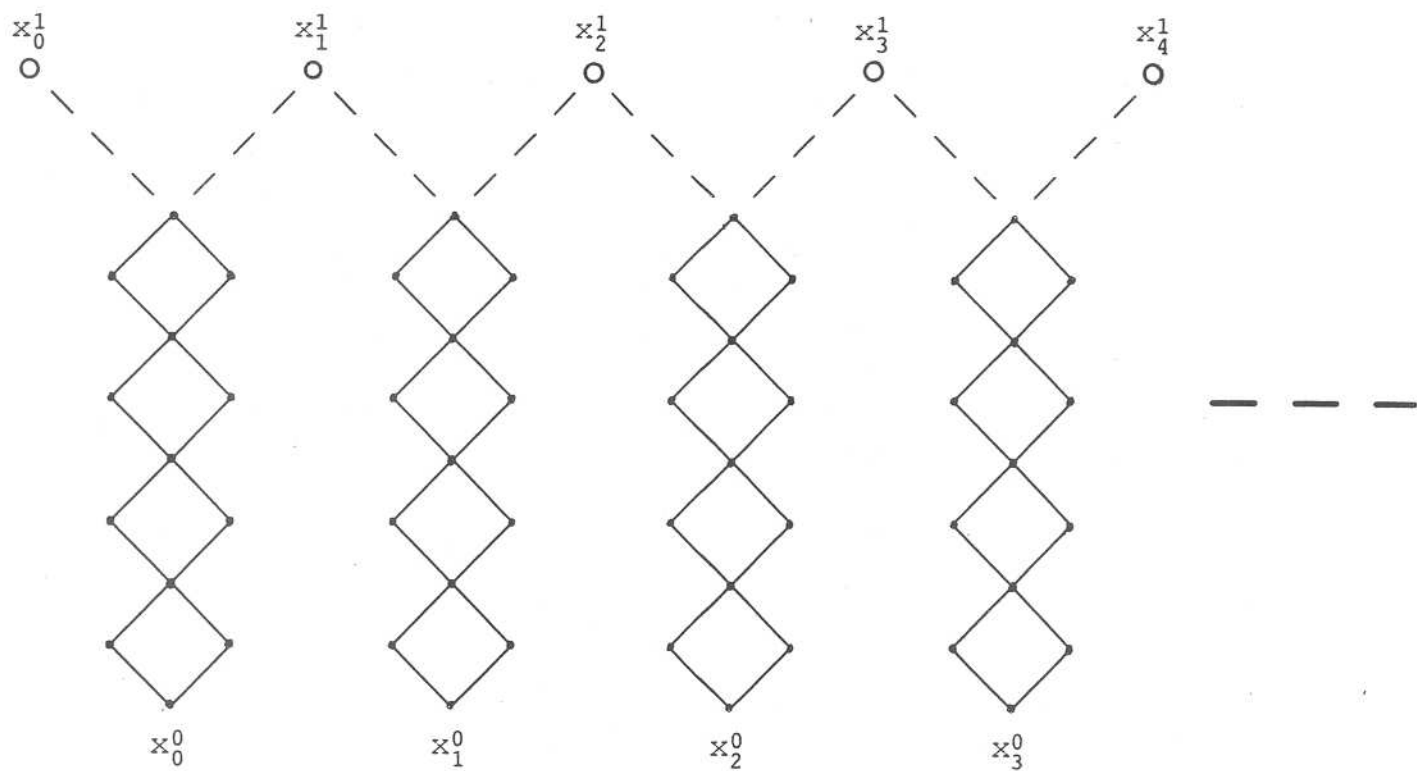
Figure 3. A 1-graph G^1 in which walks are needed to define distances between all the nodes.
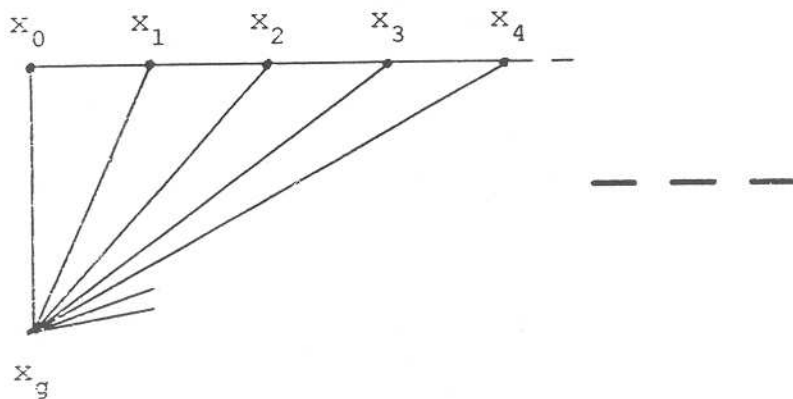
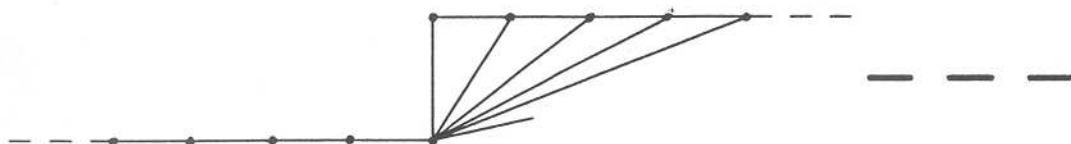Figure 1. A grounded, one-way infinite ladder. x_g is its ground node.

Figure 2. A graph G consisting of a grounded, one-way infinite ladder L and a one-ended path connected to L only at its ground node, as shown.